The set theoryofication of mathematics

Riley Moriss

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We have realised that theories rarely spring from the mind of one genius fully formed and complete. So it is with set theory, Cantor is neither the begining nor end of the conceptual development of set theory, however he did play the role (perhaps) of instigator and populariser and is none the less an absolutely key figure in its development.

1 Set Theory Before Cantor

These are just some remarks coming from the reading that I have done previously.

In Euclid we have the definition of a number as a "multitude composed of units". This could plausibly be interpreted as numbers as finite sets of units. As all concepts in mathematics though it is at this stage very much not the same concept of sets that we might now have. Certainly we have this notion of a multitude appearing everywhere. But is multiplicity the same thing as a set?

Now in geometry we have throughout time people talking about points in lines or on planes etc. This rings of thinking of the line as a collection of points. Despite this I dont think that this resemblence is any more than skin deep. A line is not the set of its points any more than a person is the set of its cells. The line is composed of points but it is not merely the collection of them. The whole is greater than the sum of its parts. Apparently this point is made "The line is composed of points" is precisely something Aristotle denies: in Physics VI.1, he argues that a continuum cannot be composed of indivisibles

In the 19th century we have Cauchy and Galois working with the collection of permutations. Galois calls them groups, in the sense of a group of people a collection. Cauchy even refers to "the set of values that a function may take". In this work it is very clear that what is being considered really is a collection or set, and that amongs them the elements enjoy some extra relations, but that there is

not really much more stucture. Finally in the later part of this century sets are really starting to come around in the work of Weirstrass and Dedekind, as their definitions of the reals are explicitly as sets of rationals. Dedekind also developed elementary set theory a little bit in his "What is a number and what is it for?".

Dedekinds number paper is not right, but his other work is relevant. Bolzano 'Paradoxes of the Infinite (1851)', Boole 'The Laws of Thought (1854)'.

2 Dedekinds Number Theory

In this period Dedekind, Cantor, Heine and Meray all came up with essentially the same definition of real number, as sets of rationals. Both Dedekind and Cantor were absolutely key to the furtherance of set theory. First Dedekind developed the beginnings of finite set theory in "What are numbers and what are they for?" moreover he also had used the same concept to make Kummers ideal numbers rigorous. Dedekind had defined ideal numbers as collections of numbers, these were ideals in the modern sense. Note that Kronecker had an alternative formulation of ideal numbers, and Gray posits that it is largely due to the support of Hilbert that Dedekinds theory came down to us. Cantor on the other hand had discovered properly infinite sets and in his theory of cardinal arithmetic had found a new system of numbers. This was highly controversial, but the key point is that it led to an investigation of the foundation of numbers and arithmetic, as all the usual operations straightforwardly extend to transfinite ordinals. Gray puts it as follows It is therefore amusing, and typical of the modernist shift, that just as two theories of real numbers were proposed that removed doubt from the foundations of the calculus differences of opinion opened up about the integers. The highly contentious claim to have discovered transfinite numbers required at the very least a clear account of finite numbers, so that the new ones could legitimately be seen as sufficiently numberlike to merit their name. Cantors proof of the existence of a bijection between the unit interval and the unit square provided further issues for the foundations of geometry as it called into question the notion of dimension. On the other hand Dedekind (and Cantor to an extent) had a parallel theory in which sets formed the foundation, they were the "immediate products of pure thought" and thus could in fact form the basis of a logicist program. Remark. Cantor is much painted as an outcast, in fact many people, including Poincare, were supporters of his early work on the reals, sets and Fourier transforms. What was controversial was his work on transfinite cardinals and his claim that they had the same status as numbers.

3 The Work of Cantor

[CJ11] contains an extended introduction dealing with the work of Cantor and his contemporaries. The thing that he is credited with is taking sets in themselves as the object of study. Treating them as something serious and worthy of investigation, rather than just a concept that is "trivial" or just there for convenience. A full bibliography of Cantors work is available here.

Cantor gets his PhD in 1867 under Kummer and Weistrass. Thus the groundwork was layed for a man to arithmetise analysis. In 1874 he publishes "On a Property of the Collection of All Real Algebraic Numbers", which contains the proof of the non-countability of the reals, although not by a diagonal argument which would not appear until 1891. Prior to this he proved that some infinite series representations of certain functions were unique. Here he says things like "The real algebraic numbers form in their totality a set" which he then denotes (ω), where ω was a variable for a single algebraic real. Likewise he collects the totality of positive integers into a set. Acording to the wikipedia this

article was stolen from Dedekind.

Cantor developed his set theory in the six papers "Über unendliche, lineare Punktmannigfaltigkeiten" translated as "Infinite linear point manifolds" between 1879 and 1884, the fifth of which is reproduced in English in [Ewa05, Vol 2. Cantor. C]. Some of the earlier parts can be found at James Meyers blog. These earlier papers seem to focus on the establishment of ordinals and transfinite arithmetic. "my investigations in the theory of manifolds has reached a point where further progress depends on extending the concept of real integer beyond the previous boundaries... I am so dependent on this extension of the number concept that without it I should be unable to take the smallest step". According to the wikipedia it was the pursuit of the continuum hypothesis that was his main motivation. The writting is predominantly prose: "A well ordered set is a well definied set in which the elements are bounded to one another by a determinant given succession such that (i) there is a first element of the set; (ii) every single element is followed by another determinate element ..." etc. This is not the set theory we are familiar with yet, however the concept itself is clearly being treated, the notion of bijection, well ordering etc are described in prose and the details of this theory are being explicated on. There are no axioms and symbolism, the language is still philosophical "The infinite real integers ... they are concrete numbers of real significance".

The final contribution of Cantor was his two papers that make up [Can], from 1895 and 1897. This work begins with a definition of set, aggregate or 'Menge': "any collection into a whole of definite and separate objects of our intuition or our thought". The union of two sets is given the notation (A, B). The concept of cardinality is introduced as "the general concept which, by means of our active faculty of thought, arises from the aggregate when we make abstraction of the nature of its various elements and of the order in which they are given". He introduces notation for cardinalities and equivalences of cardinalities. He defines a subset by the standard definition of elements. Ordinal arithmetic is summarised and given notation. Several theorems are proven but no attempt to axiomatise is made.

To illustrate the diffence between modern set theory and Cantors set theory we might consider the existence and uniquencess of intersections.

Lemma. *Let A, B be sets then there is a unique set $A \cap B$.*

> ***Proof.*** By the axiom of seperatation we may form a set $x \in A \cap B \iff (x \in A \wedge x \in B)$. This is existence. By the axiom of extensionality any other set satisfying this will be equal, this is uniquenss.

Contrast this with Cantors treatment in 'Infinite linear point manifolds: 2' "The point set common to all of $P_1, ...$ is called the intersection and we denote it by..". One would call this an axiom if it wasnt asserted so naively. So Cantor does this with many of the concepts that today we would like to reduce and prove exist by the axioms. He is not yet formal.

4 Set Theory After Cantor

4.1 Well ordering and the first axiomatisation

The first controversy involving choice and the controversy that precipitated Zermelo to first axiomatise set theory was the 'Proof that every set can be well-ordered' (1904), available in translation [Hei02]. Cantor had conjectured in 1883 that every set could be well ordered. Several proofs and counter proofs appeared and were retracted before Zermelo presented his accepted proof.

The axiom of choice, or the idea of choice in general did not start with Zermelo. Peano had apparently rejected it in [Pea90], which is in French, asking AI to find the salient part I was given the quote "But since one cannot apply infinitely many times an arbitrary rule by which an element of a class is assigned to that class, we have constructed here a definite rule by which, under suitable hypotheses, an element of each class is assigned to it". The first use of it is cited to be by Levi in 1902, a discussion of this is given in [Mos79]. Levi gives a proof of some statement under the assumption

that some set is well ordered, he remarks "The proof may also be applied without alteration to any case.. .where we can distinguish in a univocal manner one element from each set". This seems to be an acknowledgment that at least *in this case*, choice and well ordering were interchangable.

This same paper quotes Fraenkel as describing that certain constructions of Cantor and Bernstein around 1901 were making tacit use of the axiom of choice, for instance that the countable union of countable sets is countable. It was in this environment that Zermelo found it acceptable to use in his proof. The proof proceeds by using choice to construct a 'covering' of the power set, and then shows that given a covering one can get a well ordering. "Imagine that with every subset M' there is associated an arbitrary element m that occurs in M' itself: let be called the distinguished element of M'". At the end of the short paper Zermelo remarks

> The present proof rests upon the assumption that coverings γ actually do exist, hence upon the principle that even for an infinite totality of sets there are always mappings that associate with every set one of its elements, or, expressed formally, that the product of an infinite totality of sets, each containing at least one element, itself differs from zero. This logical principle cannot, to be sure, be reduced to a still simpler one, but it is applied without hesitation everywhere in mathematical deduction. For example, the validity of the proposition that the number of parts into which a set decomposes is less than or equal to the number of all of its elements cannot be proved except by associating with each of the parts in question one of its elements.

Zermelo discusses the objections to this proof in part 2 of (1908) 'A new proof of the possibility of a well-ordering'. They range from the technical to the philosophical but come from people such as Peano and Borel as well as general concerns over the validity of set theory and arguments about well ordering due to the paradoxes. The point is that the ire of mathematics was leveled at a young mathematician and he set out to prove that the work he had done could be consistent.

The next paper in 1908 was 'Investigations in the foundations of set theory I'. His main goal there is perhaps to show that set theory with choice can be formulated without admitting the paradoxes that were plaguing mathematics at the time. Recall that this is a time before Godel and he was likely hoping to be able to actually prove the consistency of his axioms. He admits that he was not able to do this (which we now know to be a good thing) "I have not yet even been able to prove rigorously that my axioms are consistent, though this is certainly very essential; instead I have had to confine myself to pointing out now and then that the antinomies discovered so far vanish one and all if the principles here proposed are adopted". Sets for the first time in Zermelo are not merely arbitrary collections, but objects that are satisfying certain axioms; very different from the definitions of Cantor and Dedekind.

Zermelo introduces a 'domain' of 'objects', amongst which the sets will be specified as those satisfying certain axioms. The notion of equality and containment are introduced as 'certain fundamental relations'. He then begins laying out his 7 axioms (seven in number in his paper but consist of many more assertions). The axiom of extensionality is a good example of how he states these,

> If every element of a set M is also an element of N and vice versa, if, therefore, both $M \subseteq N$ and $N \subseteq M$, then always$M = N$; or, more briefiy: Every set is determined by its elements.

This is a perfectly precise prose statement given that he has defined containment and element of objects within a domain. There is notation used to make the containment and equality statement formal however it is not stated with quantifiers or logical symbols. His second axiom asserts the existence of several sets, the empty set, the singleton set $\{a\}$ for every other set a, and the paired set $\{a, b\}$ for any two sets a, b. After any axiom he dervies a couple of elementary facts before stating the next one, for instance he states that these sets are unique by extensionality.

Zermelos axiom of seperation is his third axiom, it relies of a concept of 'definite' property,

> A question or assertion is said to be definite if the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not

After stating this axiom, that given a set and a logical statement the logical statemnt defines a subset, he remarks how this axiom is the solution to the paradoxes.

> By giving us a large measure of freedom in defining new sets. Axiom III in a sense furnishes a substitute for the general definition of set that was cited in the introduction and rejected as untenable. It differs from that definition in that it contains the following restrictions. In the first place, sets may never be independently defined by means of this axiom but must always be separated as subsets from sets already given; thus contradictory notions such as "the set of all sets" or "the set of all ordinal numbers", and with them the "ultrafinite paradoxes", to use Hessenberg's expression, are excluded. In the second place, moreover, the defining criterion must always be definite in the sense of our definition in No. 4 (that is, for each single element x of M the fundamental relations of the domain must determine whether it holds or not), with the result that, from our point of view, all criteria such as "definable by means of a finite number of words", hence the "Richard antinomy" and the "paradox of finite denotation", vanish.

So the restriction that it carves out only subsets prevents Russells 'infinite' type paradoxes, while the definite logical expression, though loosely defined, is meant to prevent against the paradoxes of using 'unclear' logical predicates such as 'definable in under 50 words'. The next axioms are power set existence and union. With this he defines cardinal arithmetic, and proves that it satisfies certain properties. To prove that "the product of several sets can vanish only if a factor vanishes we need a further axiom" which is his Axiom 6 'The Axiom of Choice'. He includes an axiom of infinity and those are the seven. The second part of this essay is rederiving many of the theorems of Cantor etc but in this formal framework.

4.2 Fraenkel, Von Neuman and replacement

After Zermoelos axioms there was criticism from the usual philosophical suspects, Poincare, Russell, Weyl, etc. The vagueness of the seperation axiom the allowence of certain impredictive definitions and doubts that the antinomies were sufficiently removed. Fraenkel seems to be a very early adopter of the axioms, as a serious instrument. In 1922 Fraenkel publishes 'On the foundations of Cantor–Zermelo set theory', in which he asserts a clear set theoretic failure of Zermelos theory. "if one assigns to the continuum a cardinality $\aleph_0$, the existence of sets with cardinalities $\geq \aleph_\omega$, for example, cannot be proved on the basis of the axioms." Its not obvious that one would want this, but in fact it is quite natural. For instance if one starts with the integers and recursively takes the power set a countably infinite number of times then all of these sets exist, but there is no set containing all of these sets, in modern notation the following set is not provably existent $\{\mathcal{P}^i(\mathbb{N})\}_{i\in\mathbb{N}}$.

His suggested fix is the 'Axiom of Replacement': "If M is a set and each element of M is replaced by a "thing of the domain", then M passes again into a set." Given a set you can replace elements with other sets and still have a set. Thus $\{\mathcal{P}^i(\mathbb{N})\}_{i\in\mathbb{N}}$ is created by replacing the elements of $\mathbb{N}$ by the power sets. The rest of the paper is a discussion of the Zermelo axioms. This axiom he suggests is perhaps to arbitrary, "in place of the axiom cited, one that goes less far and is sharper" he suggests that replacement should be made more precise through the use of certain functions, and that this will be made more precise elsewhere, in connection to his other work in logic on 'definiteness'.

In 1928 Von Neumann in [Neu28] shows that "the replacement axiom alone, is no essential extension at all of Fraenkel's system. On the contrary: it follows from Fraenkel's remaining axioms". Von Neumann suggests an enlarged notion of replacement where more general functions are allowed to be replaced along. To be clear what Von Neumann shows is not that the full replacement axiom above, with arbitrary functions, is superflous but Fraenkles later more 'logically' function sheme replacement axiom. Hence Zermelo will still refer to Fraenkles axiom of replacement.

4.3 Zermelos second axiomatisation

These are the developments up until 1929. The theory of naive sets was replaced with an axiomatic theory of sets after some controversy around the wrong guy, his theory was however logically naive. Certain sets were not constructable in that theory and several people were trying to add axioms and construct alternate axiomatisations in which all was well. Zermelo, as well as his critics were furiously working on the concept of definitness. The suggestions about replacement had turned out to be not entirely correct, however they were gesturing at what Zermelo at least considered a real deficit. In 1930 he published "On boundary numbers and domains of sets New investigations in the foundations of set theory".

In this paper he adopts replacement, and drops the axiom of infinity for the axiom of well foundedness. He replaces the word 'definiteness' with 'propositional function' which he does not define but says he deals with elsewhere. Zermelo adopts the full Fraenkle replacement axiom quoted above almost verbatim. The rest of the paper is dedicated showing certain uniqueness of the universe theorems "a "normal domain" is determined up to isomorphic mappings by two numbers: the cardinality of its "basis", that is, the totality of its "urelements" , and its "characteristic", that is, the order type of all "basic sequences" contained in it or of all ordinal numbers represented in it by sets". Again this paper is not formulated using first order logic.

4.4 Parallel developments in logic and formalism

So we have seen that the question of 'definitness' in Zermelo spured several works on making this more precise. Von Neumann, Zermelo and Fraenkel all participated, as well as Skollem. In Zermelo 1929 "On the concept of definiteness in axiomatics" it is argued that the best solution to the problem of definitness is an axiomatic treatment, axiomatics. He first uses the language of 'models', saying "For those, too, it is uniquely decided in every model by means of the matrix of the fundamental relations whether or not they hold in it. Such a property or relation decided by means of the fundamental relations in every model is just what, in the paper mentioned above, was supposed to be called "definite"". This is meant only to be suggestive, he also proposes a 'genetic' definition

> A proposition is called "definite" for a given system if it is constructed from the fundamental relations of the system only by virtue of the logical elementary operations of negation, conjunction and disjunction, as well as quantification, all these operations in arbitrary yet finite repetition and composition.

which he is unsatisfied with because it is not intrinsic, checkable or axiomatic. He then gives his axiomatic definition. A property is definite relative to some relations on sets if it is closed under certain logical operations and set theoretic operations. This is recognisably logical, and what I would consider a precise prose statement that parallels modern treatments. Modern treatments tend to use the formal logical language, but one should note that those symbols, meanings and methods were developed at this point and so the prose formulation is a pedagogical choice, rather than a true difference.

There is a real diffence between Zermelos presentation and truly formal first order presentations and that is the axiom schema qualification. The precise setting in which one is quantifing over formulas and variables is, I think, still not fully formed in Zermelo, although it is clear in retrospect.

Summary. So far what I have tried to summarise is how sets went from a concept that was latent in all of mathematics, as people naively considered collections, to a fully axiomatised system. We have seen that collections began to be considered much more closely in 1820's by Galois and Cauchy, although still as auxillery items used to facilitate groups. Cantor and Dedekind in the 1850's, 1870's and onwards began to consider sets naively, but in of themselves, reaching its most developed notion in Cantor. The use of sets was adopted by Cantor and Bolzano to prove certain statements in analysis and cardinal arithmetic. The concept of set now more clearly explicated and more frequently used we begin to see questions of choice, in these higher cardinalities do the naive moves of finite arithmetic

and sets make sense. In addressing this Zermelo shows that choice implies all sets can be well ordered. Symultaneously the antinomies in set theory and logic proliferate at the turn of the century. This casts doubt on the work of Zermelo who is attacked to an extent by many prominent figures. Determined to show that set theory could be made rigerous, including the axiom of choice, he creates an axiomatic system in 1908 that manifestly disallows many of the antinomies. This system itself however is ambiguous, relying on notions such as 'definitness'. This along with other logical developments motivates a closer study and formulation of first order logic. Fraenkel would point out that certain large cardinals could not be constructed, and suggest an amendment to the system in 1922. Finally in 1930 the logic has developed far enough to make mostly clear what is meant by the axioms, including replacement and seperation. Thus ZFC is released fully born into the world.

5 The Adoption by Other Fields

5.1 Analysis

Analysis was the first field to be subsumed by set theory, as it was the analysts that invented set theory for this exact purpose. The next generation continued this tradition, in the hands of Borel, Baire, Frechet, Riesz, Lebesgue and others it became heavily set theoretic, with the invention of measure theory. Already in 1915 Frechet [Fr5] had considered abstract measures on arbitrary sets.

From analysis grew topology; a word that entered the zeitgeist in 1850 - 1880 with Listing who used it to distinguish naively "qualitative geometry from the ordinary geometry in which quantitative relations are treated". We want to do a more thorough investigation on the relevant works here, but a few notes from a precursory glance at the history here. It was again Frechet in 1906 who began to lay the set theoretic, abstract definitions of space in 'Sur Quelques Points du Calcul Fonctionnel' which is translated in [Wu26]. Here he writes that over the recent years there has been several generalisations on the notion of function, with the domain gradually expanding from numbers, to pairs of numbers, to lines, to spaces of ordinary functions; his goal is to give a general encompasing theory for functions defined on a 'set of any nature'. The first thing to do then is generalise the notion of continuity of functions 'The first generalization it was natural to attempt is that of the notion of continuous function'. "Now, if one wishes to extend it to operations whose variable is an element of any nature, one must first know what is to be understood by neighboring elements or by the limit of a sequence of elements. This seems impossible". His solution is essentially that of metric spaces, introducing a measure of distance between points in the domain. The details of his treatment are not yet important to us, but are largely familiar sequence based things.

In 1911 Brouwer writes 'Beweis der Invarianz der Dimensionenzahl' [Bro11], this is in the context of Cantors discovery that there were bijections between Euclidean spaces of different dimensions and Peanos discovery that there were *continuous surjections* between Euclidean spaces of different dimensions. At this stage it seems that the spaces are still being considered as subspaces of Euclidean space and continuity in terms of ϵ, δ definitions.

In 1914 Hausdorff, then in his 50's, writes a textbook on 'Basics of Set Theory' [Hau14]. Here he lovingly embraces set theory proclaiming 'Set theory is the foundation of the whole of mathematics'. He acknowledges Zermelos axiomatisation but says that 'these matters are not yet settled', so refrains from using his axioms explicitly mentioning only that "we shall here admit the naive concept of a set, but shall in fact observe the restrictions that cut off the path to that paradox". He then makes an interesting remark "One can treat a set purely as a system of its elements, without relations between these elements" but that one may also consider sets with relations between the elements, of which ordered sets form the prime example. Hausdorff notes similar to Frechet that the concept of distance can be replaced with an assignment to any sequence of elements from a set a given element of another set, to be construed as an assignment of the limit of a sequence to the sequence. Thus he has in mind 'three privlidged positions: distance, limit and neighbourhood' which all contain the same information. Hausdorff gives the definition of a metric space that we readily recognise, neighbourhoods around points

in metric spaces he writes "have a series of properties, of which we initially need only very few". Thus the definition of topological space he gives is a generalisation of the concept of neighbourhood in a metric space.

> By a topological space we understand a set E in which to the elements (points) x certain subsets $U_x \subseteq E$ are assigned, which we call neighborhoods of x, in accordance with the following...

He then gives four properties of these neighbourhoods, (A) Every point has an assigned neighbourhood, (B) Two neighbourhoods of the same point have a third neighbourhood of the same point that is contained in the original two (not necisarily the intersection), (C) any point in a neighbourhood has a neighbourhood also contained in that neihbourhood and finally (D) two distinct points have neighbourhoods that are disjoint. This is certainly more general than the metric space concept, however it is not yet the modern definiton of a topology.

Kuratowski in 1922 writes "On the Operation $\bar{A}$ in Analysis Situs" [noa]. This is an expansion on his PhD thesis dealing with the closure operation. He 'proceeds in an axiomatic fashion' assuming as given 'by assuming as given both an arbitrary set X' and an operation of 'closure' that satisfies four axioms defined on 'every subset' of X. Interestingly he refers explicitly to his logical foundations as those layed out in Couturat. In his §2 he defines a 'closed' subset to be one such that $A = \bar{A}$. He proves that the arbitrary intersection of closed sets is closed. He defines an 'open' subset such that $A = A'^{-'}$ (compliment closure compliment), and remarks that this is iff its compliment is closed, "Hence, the union of any collection of open sets is open and the intersection of two open sets is open". Note that Hausdorff had already observed that this was the case for his neighbourhood based definition. The closure operator gives an equivalent definitno to the modern topological space definition, given an arbitrary set with an abstract closure operation one obtains the set of closed (or open) subsets by taking the fixed points of the closure operator. Kuratowski has then shown that the ordinary topological axioms obtain of this thing. He however did not make this leap or definition, our search continues.

For the history after this point we consult [Moo08, §11 onwards]. The next player was Tietze in 1923 [Tie23] who defined an O-space, putting open sets in the place of pride and having axioms very similar to what we have now. Tietzes axioms are equivalent to those of Hausdorff, but structurally more familiar

> (A°) Every point x belongs to at least one open set.
> (B°) If U and V are open sets with at least one point in common, then $U \cap V$ is an open set.
> (C°) If every point x of A is a member of some open subset of A, then A is open.
> (D°) If x and y are distinct points then there is an open set U containing x and an open set V containing y such that U and V are disjoint. [1923, 294]

He further introduced the T_i seperation axioms (not under this name). Alexandrof in 1925 [Ale25] proposed his set of axioms "The purpose of this paper is to make possible the hitherto lacking, truly general, i.e. independent of the special properties of Euclidean spaces, construction of the set-theoretic topology of manifolds of arbitrary dimensions". He defines a topological space as a set in which 'certain subsets which we shall call regions' satisfy two axioms "the intersection of two regions and the union of arbitrary regions is again a region. For any two distinct points there always exists two disjoint regions containing the respective points". This only differs from the modern definition in assuming the Hausdorff condition, it is clear since every point is contained in a region that the whole space is a region and for a non-singleton set it is clear that the empty set is in the topology since I can take the intersection of any of the two disjoint opens. It is interesting to note that he is also developing a theory of nets in this paper.

After this Sierpinski introduced in 1926 a more general notion of F-space, more general even than the modern definition. It is a non-empty set E and a set of subsets called 'closed' sets such that

(1) The set E is closed.
(2) The intersection of any set of closed sets is closed.

In 1928 he proposed

(1) The empty set is an open set.
(2) K is an open set.
(3) The union of any family of open subsets of K is open. [1934,1]
(4) If p and q are distinct elements of K, then there is an open set containing p but not containing q.
(5) The intersection of two open sets is an open set. [1934, 28]

with the first three proposed seperate to the last two. Thus the story went, with different authors using different variations on these combinations. According to [Moo08] it wasnt until Bourbaki in 1940 that the current definition appeared;

> the intersection of any finite set of open sets is open, and the union of any set of open sets is open. He then noted that from these two axioms it follows that the empty set is open (as the union of the empty family of sets) and the whole space is open (as the intersection of the empty family of sets.

This definition was apparently the result of bickering amongs the Bourbakiests.

5.2 Category Theory

It is interesting to note that only 15 years after Zermelo, 5 of which were occupied by the war, category theory was introduced in 1945 [EM]. In defining a category EM use the term "aggregates", "These investigations will deal with aggregates such as a class of groups together with a class of homomorphisms". "A category is an aggregate of abstract elements called the objects". They avoid the language, or at least the term set, however they use the symbol "$\alpha_1, \alpha_2 \in \mathfrak{U}$" which suggests a certain latent relience on sets. The appendix of that paper is also dedicated to showing that every category is isomophic to a subcategory of the category of sets, which in a sense puts sets in a privelidged position.

5.3 The Establishment of Algebra

In 1870's Dedekind began discovering ideals and fields, these were however, it seems to me, merely at this time to be collections of algebraic integers, that is explicitly subsets of $\mathbb{C}$, satisfying some properties. Weber in 1893 seems to have had an influential paper that gave a definition of group and field, that would allow for a more general treatment of Galois theory [Web93], "the concept of field in particular is so framed that it is also applicable to structures which have hitherto not been included under this name, in particular the finite fields". For a group "A system of things (elements) of any kind, finite or infinite in number, becomes a group when the following assumptions are fulfilled", he then gives that there is a composition satisfying some propterties. This is essentially the definition of a set with an opertion satisfying some properties, however taking place in the naive framework of sets, possibly that of Cantor, but more likely that of before Cantor. He defines a field likewise "A group becomes a field when two kinds of composition are possible in it, of which the first is called addition, the second multiplication".

In 1914 the fiend Fraenkle that would later feature so prominantly in our story of set theory would graduate with his PhD [Fra14] "On Divisors of Zero and the Decomposition of Rings". This dissertation would provide the abstract definition of ring that would later be referenced by Noether. Before this the term ring was introduced by Hilbert in the context of polynomial rings over the integers. Fraenkels definition reads very similar to Webers for fields, using the languge of 'a system of elements'. He gives the standard axiomatic treatment. Noether continues with this definition unchanged, at least in her

1921 paper [Ber14]. Even in 1927 [Art27] Artin is talking about rings as systems of elements. To what extent they were aware of the developments in set theory, or cared is not clear from the papers themselves. Certainly Fraenkel was later intimately familar with them.

By 1930 the textbook 'Modern Algebra' by Waerden, reviewed in [GG05, §70], this first edition may be found on the internet archive, although its not clear its the first edition, I think it is, summarised the developments of Noether and Artin. According to the review "This was the first textbook of algebra systematically to present, on an abstract set-theoretic basis using axiomatic methods, the various areas of this subject as a homogeneous whole". On his use of set theory the review claims "In set theory, he forged a strictly abstract treatment of the necessary terminology and results, by adopting the standpoint of naïve set theory without becoming involved with a discussion of the fundamentals arising in this context". He does say however that "As to foundations, van der Waerden contented himself with a reference to the third edition of Einleitung in die Mengenlehre by Abraham Fraenkel" which was a text book written (first edition) in 1923 meaning 'Introduction to set theory'.

It seems fair then to say that from the begining most algebraic objects were construed as sets with extra structure and that exactly what a set meant developed exactly parallel to the theory of sets. For algebra the technicalities of cardinality were not so relevant but it did create a natural place for algebraic abstractions.

Summary. So we see that all the major areas of mathematicsm, analysis, algebra, topology, had adopted heavy set theory language and essentially absorbed it into its foundations before Zermelo had 'finalised' his axioms.

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